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Decohesive carrying capacity of a disk under tension and in-plane torsion

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Abstract

An elastic–perfectly plastic circular disk, with rigid circular inclusion of the radius a, subjected to simultaneous uniform radial tension and in-plane torsion is considered. External radius of the disk is assumed to be infinitely large, just to simplify the boundary conditions. Limit load-carrying capacity of such a disk cannot be reached, it is preceded by decohesive carrying capacity (DCC), defined by infinitely large radial strains ε_r. They attain their upper bound for $r = a$; then the shearing strains $\gamma_{r\theta}$ at $r = a$ increase infinitely as well. In contradistinction to the limit carrying capacity, the decohesive carrying capacity depends essentially on Poisson's ratio ν. Interaction curves corresponding to DCC are shown in the plane of external loadings. A comparison with the condition of discontinuous bifurcation under uniform stresses and strains is given, and full agreement is found, though in the case under consideration the state of stress is evidently non-uniform. Diagrams of stress and strain distribution at the moment of decohesion are also presented.

1. Introduction

Usually decohesion in plasticity is considered as a separate phenomenon described by various physical criteria. In real material diverse forms of decohesion may take place, either as a result of void initiation and growth, or as a result of crack initiation and propagation, or without any of these phenomena. A comparison of various physical criteria of decohesion was given e.g. by Życzkowski (1981), Clift et al. (1990), and Needleman (1994). In the paper by Clift et al. (1990) the authors stated that a limited value of unit plastic work is in the best agreement with experimental results. Plastic strains are then limited as well.

But even if we assume the idealized elastic–perfectly plastic material without any limitation of plastic strains, then decohesion may occur as a result of local infinite increase of strains or other forms of

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termination of continuous solutions and formation of inadmissible discontinuities. The corresponding loading parameter was called by Szuwalski and Życzkowski (1973) the 'decohesive carrying capacity' (DCC). This 'mathematical' criterion of decohesion may be regarded as an upper bound to all physical criteria of decohesion, since none of them admits infinitely large strains. Many papers devoted to DCC were reviewed by Szuwalski (1990). From among more recent papers on this topic we mention that by Życzkowski and Tran (1997), devoted to a cylindrical shell under combined loading.

In order to ensure internal consistency of the above-mentioned criterion of decohesion, connected with infinite increase of normal strains, like $\partial u/\partial x \to \infty$, one should employ a finite-strain theory. Such an approach was used by Życzkowski and Szuwalski (1982), Skrzypek and Życzkowski (1983), Szuwalski and Życzkowski (1984), Życzkowski et al. (1992). The relevant analysis, much more complicated, changes the criterion of decohesion from $\partial u/\partial x \to \infty$ to the type $\partial \sigma/\partial x \to \infty$, but for positive stresses and strains numerical values of DCC remain almost without change.

Parallelly, or even earlier, elastic–plastic processes leading to discontinuities in velocities across a certain characteristic surface were studied. It was shown by Hill (1958), Rudnicki and Rice (1975), Rice (1976), that this phenomenon may be described as a type of loss of material stability and simultaneously as the loss of ellipticity of governing differential equations, namely formation of discontinuities is due to bifurcation of equilibrium states. A shorter, though not quite precise name 'discontinuous bifurcation' is now in common use. Ottosen and Runesson (1991), Neilsen and Schreyer (1993) discussed properties of discontinuous bifurcation solutions for an elastic–plastic body by means of spectral analysis in general three-dimensional case; Runesson et al. (1991) analyzed in detail plane stress and plane strain states. Życzkowski (1999) performed a similar analysis for the Burzyński–Torre paraboloidal yield condition. The states of stress and strain were assumed to be uniform (homogeneous) and direction of discontinuity was found by maximization of the hardening modulus with respect to directional cosines.

Discontinuous bifurcations are directly connected with the decohesion phenomena. This was mentioned by Thomas (1957, 1961), and considered in detail by Schreyer and Zhou (1995). They noticed that at discontinuous bifurcation there exists a situation on the verge of loss of compatibility leading to decohesion. Then either a sudden decohesion takes place, as assumed by Szuwalski and Życzkowski (1973), or a certain postcritical analysis is possible if one introduces a certain relation between the traction force across the discontinuity line and displacement jump, Needleman (1987), Mróz and Kowalczyk (1989), Skrzypek (1993). Usually the loading parameter in postcritical states decreases, and hence the term 'decohesive carrying capacity' for the loading corresponding to discontinuous bifurcation is justified. More essential differences may be due to non-uniformity of the stress state, since then the surface or line of discontinuity may be a priori prescribed (for example by the line of local maxima or suprema of strain intensity or of another appropriate strain or stress invariant) and not subject to evaluation as it is usual in the theory of discontinuous bifurcations. Various attempts to admit the solutions showing some kinds of discontinuities are connected mainly with variational approach; we mention here the papers by Temam and Strang (1980), Anzellotti and Giaquinta (1980, 1982), Seregin (1985), and Repin (1991, 1994).

Until now, all the papers devoted to decohesive carrying capacity were restricted to normal stresses only, it means to the cases of principal stresses and known principal directions. In the present paper we consider a disk with rigid circular inclusion (for example a rigid shaft perpendicular to the disk), subjected to simultaneous radial tension and in-plane torsion. The external radius of the disk is assumed to tend to infinity; such an assumption simplifies essentially the boundary conditions but does not introduce major qualitative changes when comparing to a disk with finite external radius. Szuwalski (1979) analyzed that problem for pure radial tension. Pure in-plane torsion was analyzed by Seregin (1984), and Repin (1994); they found collapse due to tangential slip. The main purpose of the present paper is to construct the interaction curves corresponding to DCC for the combined case under consideration and to compare the criterion of decohesion with that derived by Runesson et al. (1991) for discontinuous bifurcation in this case. In view of non-homogeneous state of stress these criteria might be different; however, it will be shown that in the case under consideration they coincide.

Basic assumptions of the paper are as follows:

1. The material is elastic–perfectly plastic with Young's modulus E and yield-point stress in uniaxial tension σ_0, with elastic compressibility described by Poisson's ratio v, subject to the Huber–Mises–Hencky (HMH) yield condition.
2. The analysis is restricted to small strains.
3. The Hencky–Ilyushin deformation theory is employed (similarity of stress and strain deviators); for proportionally increasing external loadings it may be regarded as justified. The Prandtl-Reuss theory would essentially complicate the calculations, but—in view of a comparison performed by Szuwalski and Życzkowski (1973) for pure tension—only minor changes in results may be expected.
4. The disk of infinite dimensions has a rigid immovable circular inclusion of radius a and is subjected

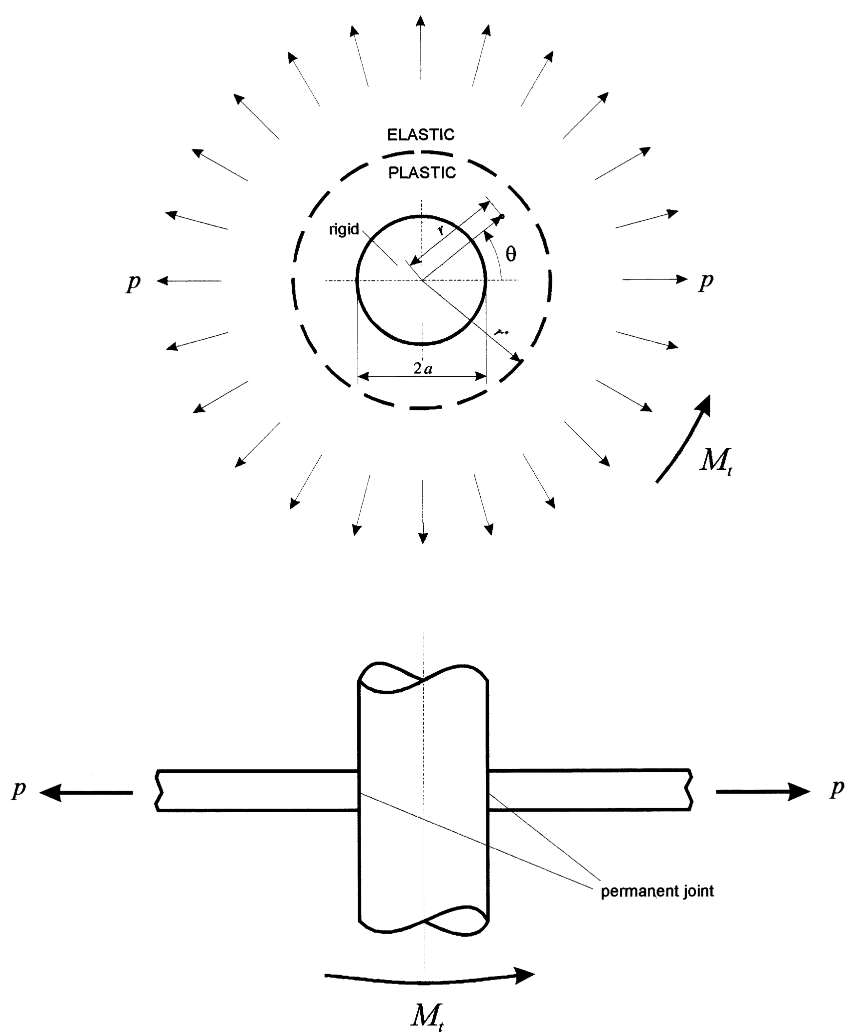

Fig. 1. Disk with rigid inclusion under radial tension and in-plane torsion.

to radial tension p and in-plane twisting moment M_t (Fig. 1).

5. Circularly symmetric deformations of the disk will only be considered. Possible loss of circular symmetry in cylinders was considered by Skrzypek and Życzkowski (1969), Storakers (1971), and in disks—by Tvergaard (1978), but such effects will be neglected. Cylindrical coordinates r, θ, z will be employed.
6. The decohesive carrying capacity of the disk will be defined by infinite increase of radial strains ε_r at $r = a$; shearing strains $\gamma_{r\theta}$ will then increase infinitely as well, whereas circumferential strains $\varepsilon_\theta = 0$ in view of the boundary condition.

2. Governing equations in the elastic–plastic range

In view of circular symmetry of the disk, $\sigma_{ij} = \sigma_{ij}(r)$, the equations of internal equilibrium, valid both in the elastic and in the plastic zone, take the form

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{1}$$

$$\frac{d\tau_{r\theta}}{dr} + 2\frac{\tau_{r\theta}}{r} = 0. \tag{2}$$

The distribution of shearing stresses is statically internally determinate, namely, integrating (2) we obtain

$$\tau_{r\theta} = \frac{C_\theta}{r^2}, \tag{3}$$

where C_θ is a constant. Calculating the in-plane twisting moment M_t we find

$$M_t = \int_0^{2\pi} \tau_{r\theta} r^2 \, d\theta = 2\pi C_\theta, \tag{4}$$

and hence, in both zones,

$$\tau_{r\theta} = \frac{M_t}{2\pi r^2}. \tag{5}$$

For sufficiently small values of external loadings the whole disk is elastic and we arrive at the general Lamé's solution with superposed torsion:

$$\sigma_r = A + \frac{B}{r^2}, \quad \sigma_\theta = A - \frac{B}{r^2}, \tag{6}$$

$$u_r = \frac{1}{E}\left[(1-\nu)Ar - (1+\nu)\frac{B}{r}\right], \quad u_\theta = Cr - \frac{(1+\nu)M_t}{2\pi E r}, \tag{7}$$

$$\varepsilon_r = \frac{(1-\nu)A}{E} + \frac{(1+\nu)B}{Er^2}, \quad \varepsilon_\theta = \frac{(1-\nu)A}{E} - \frac{(1+\nu)B}{Er^2}, \quad \gamma_{r\theta} = \frac{(1+\nu)M_t}{\pi E r^2}. \tag{8}$$

The boundary conditions

$$\sigma_r(\infty) = p, \quad u_r(a) = 0, \quad u_\theta(a) = 0, \tag{9}$$

yield

$$A = p, \quad B = \frac{1-\nu}{1+\nu} p a^2, \quad C = \frac{(1+\nu) M_t}{2\pi E a^2}. \tag{10}$$

The HMH yield condition in the case under consideration has the form

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_r \sigma_\theta + 3\tau_{r\theta}^2 = \sigma_0^2, \tag{11}$$

and substituting (5), (6) and (10) into this condition we obtain the equation of the elastic interaction curve (boundary of the elastic range)

$$\frac{4\left(1-\nu+\nu^2\right)}{(1+\nu)^2} p^2 + \frac{3}{4\pi^2 a^4} M_t^2 = \sigma_0^2. \tag{12}$$

If the loadings p and M_t exceed (12), in general a plastic zone $a < r < r^*$ will appear (except in the case of an incompressible material, $\nu = \frac{1}{2}$, which will be discussed later). In the plastic zone we have three equations (1), (2), and (11) for three unknown stress components σ_r, σ_θ and $\tau_{r\theta}$, and the problem seems also to be internally statically determinate, but in view of the boundary conditions in displacements, (9), the strains and displacements must be calculated anyway. But first we can derive differential equation for stresses, and then, separately, for strains and displacements. General solution for circularly symmetric perfectly plastic states of disks was derived by Życzkowski (1958) for arbitrary yield condition, nonhomogeneity, thermal loading and body forces, but here, in view of the HMH yield condition (11), we shall go some other, more convenient way. Namely, in view of (5), we parametrize the yield condition as follows

$$\sigma_r = \frac{2}{\sqrt{3}} \sqrt{\sigma_0^2 - \frac{3M_t^2}{4\pi^2 r^4}} \sin\zeta,$$

$$\sigma_\theta = \frac{2}{\sqrt{3}} \sqrt{\sigma_0^2 - \frac{3M_t^2}{4\pi^2 r^4}} \sin\left(\zeta + \frac{\pi}{3}\right), \tag{13}$$

thus replacing two unknown stresses by one dimensionless parameter ζ, which is now determined by eqn (1). The substitution (13) may be regarded as a generalization of the well-known Nadai–Sokolovsky parametrization of the HMH yield condition. Related problems of elastic–plastic disks under combined tension and in-plane torsion were initiated by Parasyuk (1948), Nordgren and Naghdi (1963), and numerous references are given by Życzkowski (1981), but none of those papers has considered decohesive carrying capacity.

After some rearrangements we obtain for ζ the following first-order differential equation

$$r\frac{d\zeta}{dr} \cos\zeta + \sin\left(\zeta - \frac{\pi}{3}\right) + \frac{6M_t^2}{4\pi^2 \sigma_0^2 r^4 - 3M_t^2} = 0. \tag{14}$$

It is seen that eqn (14) is singular for $\zeta = (\pi/2)$, and just this case will be considered in detail. So, it is more convenient to regard r as dependent, and ζ as independent variable. Moreover, for numerical integration we introduce dimensionless quantities, and namely dimensionless twisting moment m and dimensionless radius ρ,

$$m = \frac{M_t\sqrt{3}}{2\pi\sigma_0 a^2}, \quad \rho = \frac{r}{a}. \tag{15}$$

The coefficients are chosen in such a way as to ensure $m \leqslant 1$, $\rho \geqslant 1$, since $m = 1$ corresponds not only to elastic carrying capacity under pure torsion, as in (12), but also to maximal value of the moment which may be transmitted by the elastic–perfectly plastic body under consideration. Eqn (14) will be rewritten in the form

$$\frac{d\rho}{d\zeta} = F_1(\rho, \zeta), \tag{16}$$

where

$$F_1(\rho, \zeta) = -\frac{\left(\rho^4 - m^2\right)\rho \cos\zeta}{\left(\rho^4 - m^2\right)\sin\left(\zeta - \frac{\pi}{3}\right) + 2m^2}. \tag{17}$$

The Hencky–Ilyushin deformation theory has the form

$$e_{ij} = \varphi s_{ij}, \tag{18}$$

where e_{ij} denote deviatoric strains, and s_{ij} deviatoric stresses, respectively, and φ is the variable plastic modulus to be determined. Combining (18) with compatibility equation

$$\varepsilon_r = \varepsilon_\theta + r\frac{d\varepsilon_\theta}{dr}, \tag{19}$$

we find for φ the formula

$$\varphi = \frac{r}{\sigma_r - \sigma_\theta}\frac{d\varepsilon_\theta}{dr}. \tag{20}$$

Now, using once more eqn (18) combined with the law of volume change for mean strain ε_m and σ_m,

$$\varepsilon_m = \frac{1-2v}{E}\sigma_m, \tag{21}$$

we derive the following equation for ε_θ:

$$r\frac{d\varepsilon_\theta}{dr} - 3\frac{\sigma_r - \sigma_\theta}{2\sigma_\theta - \sigma_r}\varepsilon_\theta + \frac{1-2v}{E}\frac{\sigma_r^2 - \sigma_\theta^2}{2\sigma_\theta - \sigma_r} = 0. \tag{22}$$

Though ε_θ is dimensionless, it will be more convenient to introduce another dimensionless strain by formula

$$\tilde{\varepsilon}_\theta = \frac{E}{\sigma_0}\varepsilon_\theta, \tag{23}$$

since then the number of constants is reduced; $\tilde{\varepsilon} = 1$ at the yield-point stress in uniaxial tension. Regarding consistently ζ as independent variable we introduce the derivative $d\tilde{\varepsilon}_\theta/d\zeta$ by using (16) and (17). Finally, substituting (13) we obtain the equation in the following dimensionless form

$$\frac{\mathrm{d}\tilde{\varepsilon}_\theta}{\mathrm{d}\zeta} = F_2(\rho, \zeta, \tilde{\varepsilon}_\theta), \tag{24}$$

where

$$F_2(\rho, \zeta, \tilde{\varepsilon}_\theta) = \frac{(\rho^4 - m^2)\sqrt{3}\sin\left(\zeta - \frac{\pi}{3}\right)}{(\rho^4 - m^2)\sin\left(\zeta - \frac{\pi}{3}\right) + 2m^2}\left[\frac{2(1-2v)}{3\rho^2}\cos\left(\zeta - \frac{\pi}{3}\right)\sqrt{(\rho^4 - m^2)} - \tilde{\varepsilon}_\theta\right], \tag{25}$$

and $\rho = \rho(\zeta)$ is determined by (16) and (17).

Numerical integration of (16) and (24) makes it possible to determine ρ, ζ and $\tilde{\varepsilon}_\theta$ in the plastic zone $1 < \rho < \rho^*$. Knowing these quantities we can evaluate the remaining unknowns. From (18) and (21) we find

$$\varepsilon_\mathrm{r} = \frac{1}{2\sigma_\theta - \sigma_\mathrm{r}}\left[(2\sigma_\mathrm{r} - \sigma_\theta)\varepsilon_\theta - \frac{1-2v}{E}\left(\sigma_\mathrm{r}^2 - \sigma_\theta^2\right)\right]. \tag{26}$$

Introducing $\tilde{\varepsilon}_\mathrm{r}$ and $\tilde{\gamma}_{\mathrm{r}\theta}$ by the formulae of the type (23) and substituting (13) we obtain

$$\tilde{\varepsilon}_\mathrm{r} = \frac{1}{\cos\zeta}\left[\tilde{\varepsilon}_\theta \sin\left(\zeta - \frac{\pi}{6}\right) + \frac{1-2v}{\rho^2\sqrt{3}}\sqrt{\rho^4 - m^2}\cos\left(2\zeta - \frac{\pi}{6}\right)\right], \tag{27}$$

$$\varphi E = \frac{1}{\cos\zeta}\left[\frac{3\rho^2\tilde{\varepsilon}_\theta}{2\sqrt{\rho^4 - m^2}} - (1-2v)\sin\left(\zeta + \frac{\pi}{6}\right)\right], \tag{28}$$

$$\tilde{\gamma}_{\mathrm{r}\theta} = \frac{2\varphi E}{\sigma_0}\tau_{\mathrm{r}\theta} = \frac{m}{\sqrt{3}\cos\zeta}\left[\frac{3\tilde{\varepsilon}_\theta}{\sqrt{\rho^4 - m^2}} - \frac{2(1-2v)}{\rho^2}\sin\left(\zeta + \frac{\pi}{6}\right)\right]. \tag{29}$$

Radial displacements are determined simply by the formula $u_\mathrm{r} = r\varepsilon_\theta$. Geometrical relation for circumferential displacements can be written in the form

$$\gamma_{\mathrm{r}\theta} = r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\theta}{r}\right), \tag{30}$$

and hence, in dimensionless form,

$$\tilde{u}_\theta \overset{\mathrm{def}}{=} \frac{E}{a\sigma_0}u_\theta = \rho\int_1^\rho \frac{\tilde{\gamma}_{\mathrm{r}\theta}(\bar{\rho})}{\bar{\rho}}\,\mathrm{d}\bar{\rho}, \tag{31}$$

where $\bar{\rho}$ is the variable of integration, and the boundary condition $\tilde{u}_\theta(1) = 0$ is already taken into account. Of course, an integration over ζ can also be introduced via (16).

In order to obtain effective solution for an elastic–plastic disk under radial tension at infinity p and twisting moment M_t, we need eight boundary conditions besides that accounted in (31). Namely, we need two boundary conditions to start integration of (16) and (24), and we have to evaluate six constants: ζ_a corresponding to $r = a$ or $\rho = 1$, radius of elastic–plastic interface ρ^*, the corresponding value of ζ denoted by ζ^*, and three constants A, B, C in the general elastic solution (6) and (7). These boundary conditions look as follows:

at the rigid inclusion

$$\tilde{\varepsilon}_{\theta}^{(\rho)}(\zeta_{\mathrm{a}}) = 0, \quad \rho^{(\rho)}(\zeta_{\mathrm{a}}) = 1, \tag{32}$$

at the elastic–plastic interface

$$\sigma_{\mathrm{r}}^{(\mathrm{p})}(\zeta^*) = \sigma_{\mathrm{r}}^{(\mathrm{e})}(\zeta^*), \quad \tau_{\mathrm{r}\theta}^{(\mathrm{p})}(\zeta^*) = \tau_{\mathrm{r}\theta}^{(\mathrm{e})}(\zeta^*),$$

$$u_{\mathrm{r}}^{(\mathrm{p})}(\zeta^*) = u_{\mathrm{r}}^{(\mathrm{e})}(\zeta^*), \quad u_{\theta}^{(\mathrm{p})}(\zeta^*) = u_{\theta}^{(\mathrm{e})}(\zeta^*),$$

$$\sigma_{\mathrm{e}}^{(\mathrm{e})}(\zeta^*) = \sigma_0, \quad \text{or equivalently} \quad \varphi(\zeta^*) = \frac{1}{2G} = \frac{1+v}{E}, \tag{33}$$

and at infinity

$$\sigma_{\mathrm{r}}^{(\mathrm{e})}(\infty) = p, \tag{34}$$

where the superscripts (e) and (p) refer to the elastic and plastic zone, respectively, and σ_{e} denotes the HMH effective stress.

Formally, the initial conditions (32) make it possible to integrate numerically the eqns (16) and (24), but, in the general elastic–plastic problem the value of the parameter ζ_{a} remains unknown. It should be adjusted by 'shooting' procedure in such a way as to obtain the prescribed value of the loading p (the second loading m appears in the equations and should simply be substituted). So, first we determine the possible interval for ζ_{a} to facilitate the initial choice,

At the beginning of the elastic–plastic process (elastic carrying capacity) the stresses σ_{r} and σ_{θ} at $r = a$ are determined simultaneously by eqns (6) with substituted (10) and by eqns (13):

$$\frac{2}{1+v}p = \frac{2}{\sqrt{3}}\sqrt{\sigma_0^2 - \frac{3M_{\mathrm{t}}^2}{4\pi^2 a^4}}\sin\zeta_{\mathrm{a}},$$

$$\frac{2v}{1+v}p = \frac{2}{\sqrt{3}}\sqrt{\sigma_0^2 - \frac{3M_{\mathrm{t}}^2}{4\pi^2 a^4}}\sin\left(\zeta_{\mathrm{a}} + \frac{\pi}{3}\right). \tag{35}$$

Dividing side by side we obtain

$$\frac{1}{v} = \frac{2\sin\zeta_{\mathrm{a}}}{\sin\zeta_{\mathrm{a}} + \sqrt{3}\cos\zeta_{\mathrm{a}}}, \tag{36}$$

and finally

$$\tan\zeta_{\mathrm{a}} = -\frac{\sqrt{3}}{1-2v}. \tag{37}$$

Hence, for a compressible material, $0 \leqslant v < 1/2$, the parameter ζ_{a} starts from a value lying in the interval $\pi/2 < \zeta_{\mathrm{a}} \leqslant 2\pi/3$. It turns out that with increasing loadings this parameter decreases, but the above interval holds for the whole elastic–plastic range.

Particular attention should be paid to the incompressible material, $v = 1/2$. Then the integral of (24) with the condition (32) is simply $\varepsilon_{\theta} \equiv 0$, and from (27) $\varepsilon_{\mathrm{r}} \equiv 0$. This is in obvious contradiction with basic assumptions of plastic deformations, hence the plastic zone cannot develop and the elastic carrying capacity (12) terminates the process.

From the condition (34) we obtain simply $A = p$, and the second of the conditions (33) is satisfied automatically in view of (5) valid in both zones. The remaining four continuity conditions will be written in the dimensionless form

$$\frac{2}{\sqrt{3}}\sqrt{1-\frac{m^2}{\rho^{*4}}}\sin\zeta^* = q + \frac{B}{\sigma_0 a^2 \rho^{*2}}, \tag{38}$$

$$\tilde{\varepsilon}^{*(\mathrm{p})}_{\theta} = (1-\nu)q - \frac{(1+\nu)B}{\sigma_0 a^2 \rho^{*2}}, \tag{39}$$

$$\rho^* \int_1^{\rho^*} \frac{\tilde{\gamma}^{(\mathrm{p})}_{\mathrm{r}\theta}(\bar{\rho})}{\bar{\rho}}\mathrm{d}\bar{\rho} = \frac{CE}{\sigma_0}\rho^* - \frac{(1+\nu)m}{\sqrt{3}\rho^*}, \tag{40}$$

$$q^2 + 3\frac{B^2}{\sigma_0^2 a^4 \rho^{*4}} + \frac{m^2}{\rho^{*4}} = 1, \tag{41}$$

where $q = p/\sigma_0$ denotes dimensionless radial loading. Condition (40) determines directly the constant C, whereas (38), (39) and (41) determine B and q and give a relation between $\tilde{\varepsilon}^{*(\mathrm{p})}_{\theta}$, ζ^* and ρ^* which determines the elastic–plastic interface, it means the end-point of numerical integration of (16) and (24). Substituting B evaluated from (38) into (41) we obtain a quadratic equation for q with the roots

$$q = \sqrt{1-\frac{m^2}{\rho^{*4}}}\left(\frac{\sqrt{3}}{2}\sin\zeta^* \pm \frac{1}{2}\cos\zeta^*\right). \tag{42}$$

Considering continuity of solutions at the elastic carrying capacity, $\zeta^* = \zeta_{\mathrm{a}}$, we find that in (42) upper sign should be taken, and we rewrite this formula as follows

$$q = \sqrt{1-\frac{m^2}{\rho^{*4}}}\cos\left(\zeta^* - \frac{\pi}{3}\right). \tag{43}$$

This is a direct generalization of the formula derived by Szuwalski and Życzkowski (1973) for the case $m = 0$. We further obtain

$$\frac{B}{\sigma_0 a^2 \rho^{*2}} = \sqrt{1-\frac{m^2}{\rho^{*4}}}\left(\frac{1}{2\sqrt{3}}\sin\zeta^* - \frac{1}{2}\cos\zeta^*\right), \tag{44}$$

$$\tilde{\varepsilon}^{*(\mathrm{p})}_{\theta} = \sqrt{1-\frac{m^2}{\rho^{*4}}}\left(\frac{1-2\nu}{\sqrt{3}}\sin\zeta^* + \cos\zeta^*\right). \tag{45}$$

So, eqns (16) and (24) should be integrated numerically up to the value $\zeta = \zeta^*$ at which (45) is satisfied, and then q is determined by (43). The unknown parameter ζ_{a} should then be chosen so as to obtain for q the prescribed value of external loading.

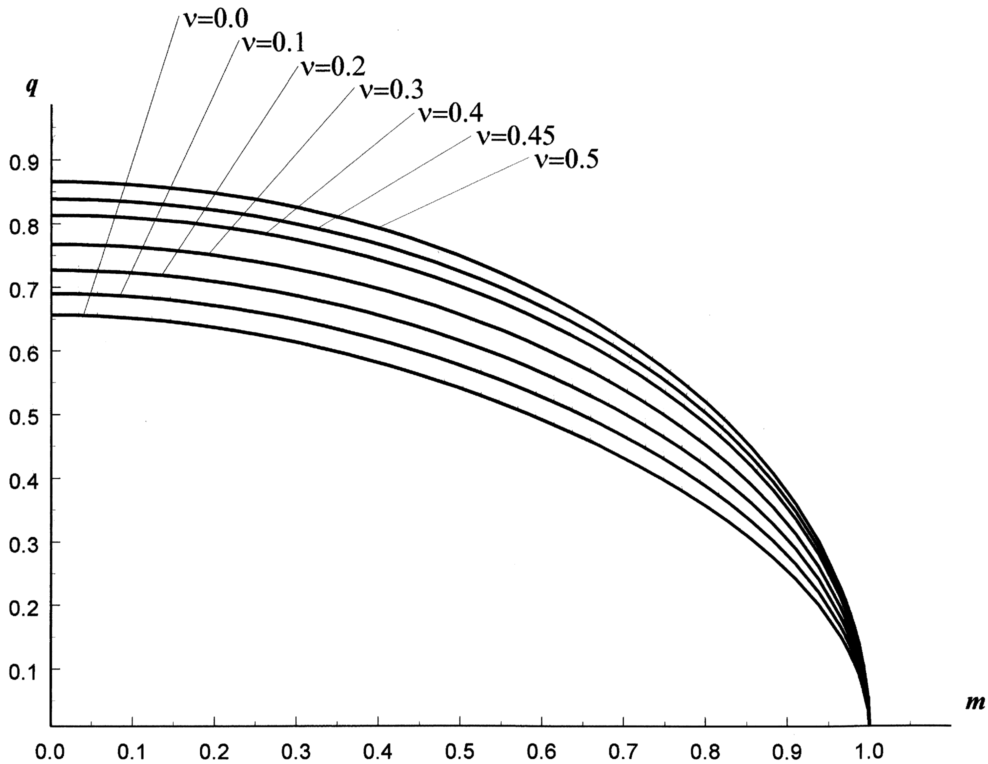

Fig. 2. Interaction curves corresponding to decohesive carrying capacity.

3. Decohesive carrying capacity

In the elastic–plastic range the value of the parameter ζ_a starts from that given by (37) and then it decreases. The deformation process terminates if ζ_a approaches $\pi/2$: indeed, then at the clamped boundary $\varepsilon_r \to \infty$, $\varphi \to \infty$, $\gamma_{r\theta} \to \infty$ and the process cannot be continued. According to the definition adopted, the decohesive carrying capacity of the disk is then exhausted.

Numerical evaluation of the DCC is even simpler than the elastic–plastic analysis. The initial conditions at the rigid inclusion, (32), now take the form

$$\tilde{\varepsilon}_\theta^{(p)}(\pi/2) = 0, \quad \rho^{(p)}(\pi/2) = 1, \tag{46}$$

and numerical integration of (16) and (24) for a given value of m, $m < 1$, starts without any unknown parameter. It goes up to the point $\zeta = \zeta^*$ at which (45) is satisfied. Then also ρ^* is determined and (43) gives directly the value of q, denoted here by $\hat{q}$.

Interaction curves in the plane $m - q$, corresponding to decohesive carrying capacity of the disk under consideration, depend essentially on Poisson's ratio ν, it means on the compressibility of the material. They are shown in Fig. 2. for various values of ν. They are convex; this is rather an exception, since in many cases interaction curves corresponding to DCC exhibit concavities, Szuwalski and Życzkowski (1984), Skrzypek and Muc (1988), Bielski and Skrzypek (1989), Życzkowski et al. (1992), Życzkowski and Tran (1997).

The difference between the interaction curves corresponding to elastic carrying capacity and DCC

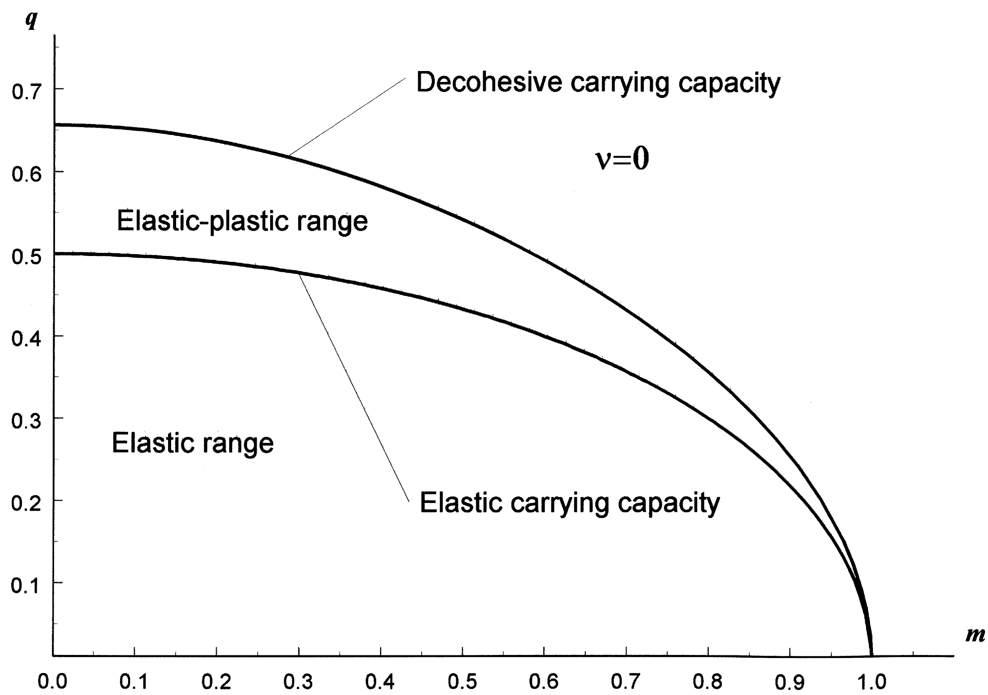

Fig. 3. Ranges of deformation for $\nu = 0$.

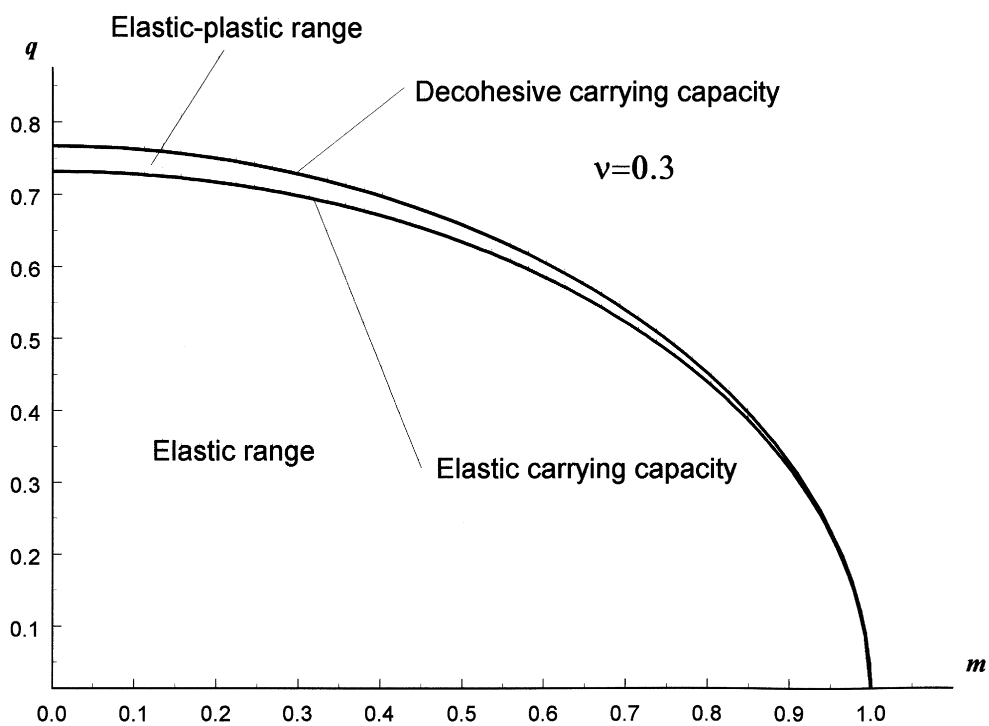

Fig. 4. Ranges for deformation for $\nu = 0.3$.

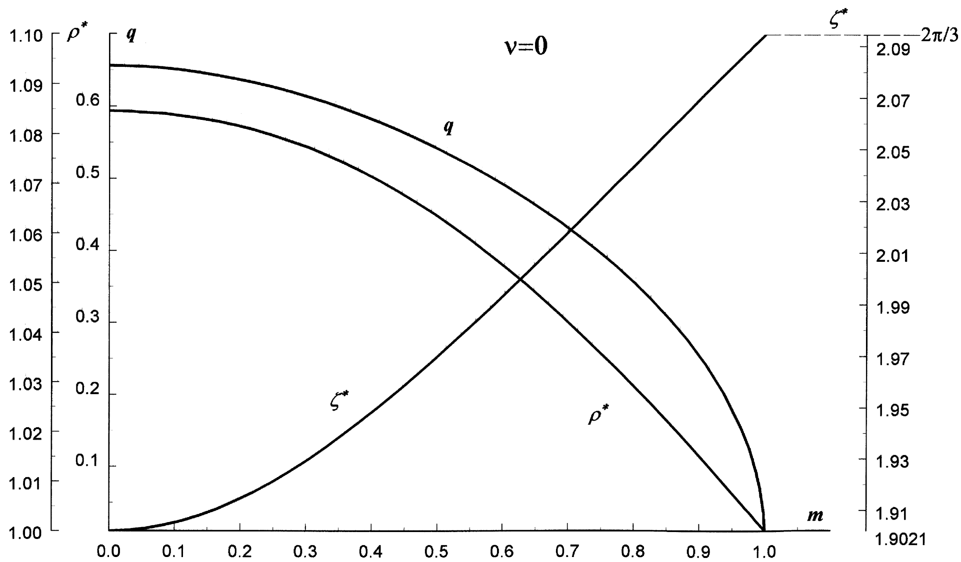

Fig. 5. Dependence of ρ^* and ζ^* in terms of m and q for $v = 0$.

depends also on v. Figs. 3 and 4 show both curves for $v = 0$ and $v = 0.3$, whereas for $v = 0.5$ both these curves coincide and the elastic–plastic range vanishes. Moreover, Fig. 5 gives the relations $\rho^* = \rho^*(m)$ and $\zeta^* = \zeta^*(m)$ for $v = 0$ and the DCC curve making it possible to evaluate also $\rho^* = \rho^*(q)$ and $\zeta^* = \zeta^*(q)$.

In order to analyze the order of singularity of $\tilde{\varepsilon}_r$ at DCC we expand the solutions of (16) and (24) in the vicinity of $\zeta = \pi/2$, $\rho = 1$, $\tilde{\varepsilon}_\theta = 0$ into generalized power series, and then perform suitable operations as to obtain the series $\tilde{\varepsilon}_r = \tilde{\varepsilon}_r(\rho)$, Życzkowski (1965), Feldmar and Kölbig (1986). First we introduce new variables which are small in the vicinity of the singular point:

$$\alpha = \rho - 1, \quad \beta = \zeta - \frac{\pi}{2} \tag{47}$$

Introducing them into (16) and (17), expanding $\alpha = \alpha(\beta)$ into power series and equating coefficients of consecutive powers of β at both sides we obtain

$$\alpha = \frac{1 - m^2}{1 + 3m^2}\beta^2 - \frac{2}{\sqrt{3}}\frac{(1 - m^2)^2}{(1 + 3m^2)^2}\beta^3 + \cdots. \tag{48}$$

Further, introducing (47) and (48) into (24) and (25) we obtain in similar manner

$$\tilde{\varepsilon}_\theta = \frac{1 - 2v}{1 + 3m^2}(1 - m^2)^{3/2}\beta - \frac{2(1 - 2v)(1 - 3m^2)(1 - m^2)^{3/2}}{\sqrt{3}(1 + 3m^2)^2}\beta^2 + \cdots. \tag{49}$$

In order to obtain $\tilde{\varepsilon}_\theta = \tilde{\varepsilon}_\theta(\alpha)$ we have to invert the series (48):

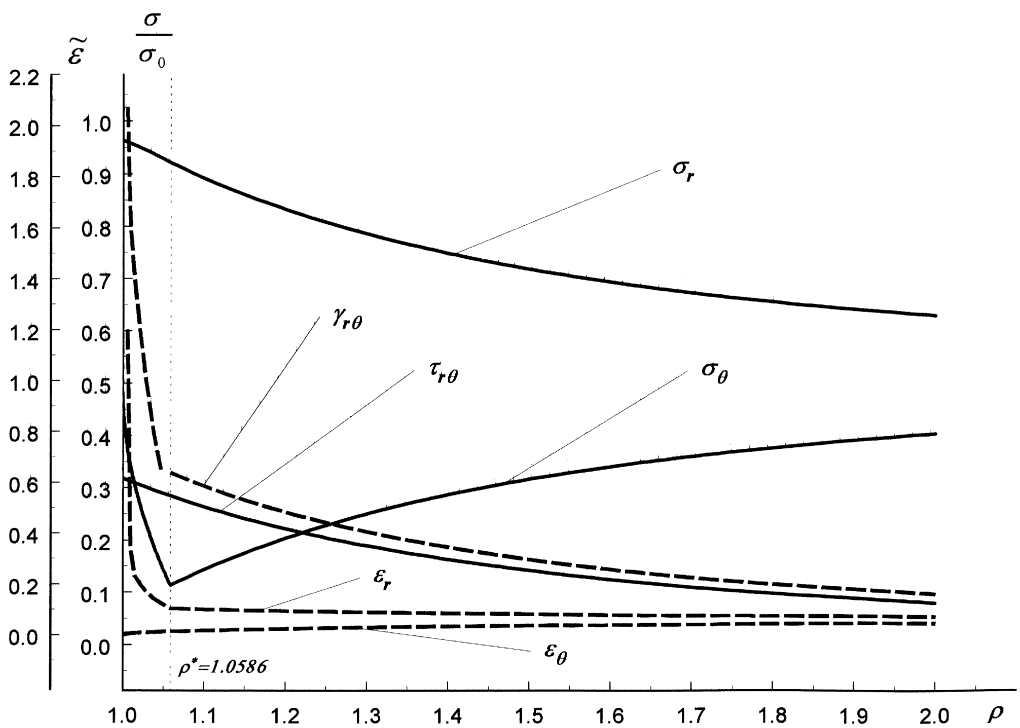

Fig. 6. Stress and strain distribution for $m = 0.5514$, $p = 0.5175$, $v = 0$.

$$\beta = \sqrt{\frac{1+3m^2}{1-m^2}}\alpha^{1/2} + \frac{\sqrt{3}}{3}\alpha + \cdots, \tag{50}$$

and substitute it into (49)

$$\tilde{\varepsilon}_\theta = \frac{(1-2v)(1-m^2)}{\sqrt{1+3m^2}}\alpha^{1/2} - \frac{(1-2v)(1-5m^2)\sqrt{1-m^2}}{\sqrt{3}(1+3m^2)}\alpha + \cdots. \tag{51}$$

Finally, from the compatibility condition (19) we obtain

$$\tilde{\varepsilon}_\mathrm{r} = \frac{(1-2v)(1-m^2)}{2\sqrt{1+3m^2}}\alpha^{-1/2} - \frac{(1-2v)(1-5m^2)\sqrt{1-m^2}}{\sqrt{3}(1+3m^2)} + \cdots. \tag{52}$$

Hence, in physical quantities, the strain $\tilde{\varepsilon}_\mathrm{r}$ for $r = a$ at decohesion tends to infinity as $(r-a)^{-1/2}$.

Expanding in a similar way the formula (29) we obtain

$$\tilde{\gamma}_{\mathrm{r}\theta} = (1-2v)m\sqrt{\frac{1-m^2}{1+3m^2}}\alpha^{-1/2} - \frac{(1-2v)m(5-m^2)}{\sqrt{3}(1+3m^2)} + \cdots \tag{53}$$

and for any non-vanishing torsion $0 < m < 1$ and $v \neq 1/2$ this strain also increases infinitely like $\tilde{\varepsilon}_\mathrm{r}$.

A diagram of stress and strain distribution in the disk at the moment of decohesion is shown in Fig. 6

for $m = 0.5514$, $p = 0.5175$ and $v = 0$; this value of v results in the largest possible plastic zone and hence the diagrams are representative.

Finally, we give a comparison of the direction of decohesion analyzed in the present paper with that of discontinuous bifurcation derived by Runesson et al. (1991). From the Mohr's circle for plane stress we find

$$\tan 2\psi_{n1} = \frac{2\tau_{r\theta}}{\sigma_r - \sigma_\theta}, \tag{54}$$

where ψ_{n1} is the angle of inclination of the normal to the direction of decohesion with respect to principal direction '1'. Substituting into (54)

$$\sigma_r = \frac{2}{\sqrt{3}}\sigma_0\sqrt{1-m^2}, \quad \sigma_\theta = \frac{1}{\sqrt{3}}\sigma_0\sqrt{1-m^2}, \quad \tau_{r\theta} = \frac{1}{\sqrt{3}}\sigma_0 m, \tag{55}$$

we obtain

$$\tan 2\psi_{n1} = \frac{2m}{\sqrt{1-m^2}}. \tag{56}$$

The same formula may be obtained from the analysis of strains, namely, if we calculate the limit of $\tilde{\gamma}_{r\theta}/\tilde{\varepsilon}_r$ for $\alpha \to 0$.

On the other hand, Runesson et al. (1991) derived the following formula for the direction of discontinuous bifurcation at plane stress:

$$\tan^2\psi_{n2} = -\frac{2\sigma_I - \sigma_{II}}{2\sigma_{II} - \sigma_I}, \tag{57}$$

where σ_I and σ_{II} are algebraically ordered principal stresses,

$$\sigma_{I,II} = \tfrac{1}{2}(\sigma_r + \sigma_\theta) \pm \tfrac{1}{2}\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau^2} \tag{58}$$

and ψ_{n2} denotes the angle between the normal to the direction of decohesion and principal direction '2', $\psi_{n2} = (\pi/2) - \psi_{n1}$.

Substituting (55) and (58) into (57) we find

$$\tan^2\psi_{n2} = \frac{\sqrt{1+3m^2} + \sqrt{1-m^2}}{\sqrt{1+3m^2} - \sqrt{1-m^2}} = \frac{\left(\sqrt{1+3m^2} + \sqrt{1-m^2}\right)^2}{4m^2}, \tag{59}$$

hence we obtain a simple formula for $\tan\psi_{n2}$, and calculating $\tan 2\psi_{n2}$ we arrive at

$$\tan 2\psi_{n2} = -\frac{2m}{\sqrt{1-m^2}}. \tag{60}$$

In view of the relation between ψ_{n1} and ψ_{n2} we find (56) and (60) in agreement.

So, in the case under consideration the directions of discontinuous bifurcation and of decohesion coincide. This result is by no means obvious, since the first (57) was derived under the assumption of uniform stress state, whereas the second, (54), for nonuniform stress state in the disk considered. For example, in slightly nonprismatic bars under simple tension such a coincidence does not take place. Hence, the coincidence of the results of (54) and (57) gives an additional argument for close relation between discontinuous bifurcation and decohesion, as was noticed by Schreyer and Zhou (1995).

4. Conclusions

1. The process of elastic–perfectly plastic deformations in a disk with rigid inclusion, subject to simultaneous radial tension and in-plane torsion, terminates with decohesive carrying capacity. After decohesion the disk is no longer in equilibrium, since the twisting moment cannot be equilibrated.
2. The decohesive carrying capacity depends essentially on Poisson's ratio v. For an incompressible material $v = 1/2$, the plastic zone cannot be developed and DCC coincides with elastic carrying capacity.
3. At the moment of decohesion radial strains ε_r increase infinitely, and the type of singularity is described by the function $(r - a)^{-1/2}$. Just for an incompressible material or pure torsion this expansion does not hold and we observe immediate collapse.
4. In the case under consideration the directions of discontinuous bifurcation and of decohesion coincide.

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References

Anzellotti, G., Giaquinta, M., 1980. Existence of displacement field for an elastoplastic body subject to Hencky's law and von Mises yield condition. Manuscripta Mathematica 32, 101–136.

Anzellotti, G., Giaquinta, M., 1982. On the existence of the fields of stresses and displacements for an elasto-perfectly plastic body in static equilibrium. Math. Pure et Appl. 61, 219–244.

Bielski, J., Skrzypek, J., 1989. Failure modes of elastic–plastic curved tubes under external pressure with in-plane bending. Int. J. Mech. Sci. 31, 435–458.

Clift, S.E., Hartley, P., Sturgess, C.E.N., Rowe, G.W., 1990. Fracture prediction in plastic deformation processes. Int. J. Mech. Sci. 32, 1–17.

Feldmar, E., Kölbig, K.S., 1986. Reduce procedures for the manipulations of generalized power series. Comp. Physics Commun. 39, 267–284.

Hill, R., 1958. A general theory of uniqueness and stability in elastic–plastic solids. J. Mech. Phys. Solids 6, 236–249.

Mróz, Z., Kowalczyk, M., 1989. Elasto-plastic post-critical analysis of disks under tension. Arch. Mech. Stos. 41, 461–480.

Needleman, A., 1987. A continuum model for void nucleation by inclusion debonding. J. Appl. Mechanics 54, 525–531.

Needleman, A., 1994. Computational modeling of material failure. Appl. Mech. Reviews 47 (6/2), 34–42.

Neilsen, M.K., Schreyer, H.L., 1993. Bifurcations in elastic–plastic materials. Int. J. Solids Structures 30, 521–544.

Nordgren, R.P., Naghdi, P.M., 1963. Loading and unloading solutions for an elastic–plastic annular plate in the state of plane stress under combined pressure and couple. Int. J. Eng. Sci. 1, 33–70.

Ottosen, N.S., Runesson, K., 1991. Properties of discontinuous bifurcation solutions in elasto-plasticity. Int. J. Solids Structures 27, 401–421.

Parasyuk, O.S., 1948. Elastic–plastic problems with non-biharmonic plastic state (in Russian). Dokl. Akad. Nauk SSSR 63, 4.

Repin, S.I., 1991. On variational formulations for discontinuous displacement fields in the problems of perfect plasticity (in Russian). Prikl. Mat. Mekh. 55, 1026–1034.

Repin, S.I., 1994. On the solutions of mathematical theory of plasticity including discontinuous displacement fields (in Russian). Prikl. Mat. Mekh. 58, 142–152.

Rice, J.R., 1976. The localization of plastic deformation. In: Koiter, W.T. (Ed.), Theoretical and Appl. Mechanics. 14th Congr. North Holland, Amsterdam, pp. 207–220.

Rudnicki, J.W., Rice, J.R., 1975. Conditions for the localization of deformation in pressure-sensitive dilatant materials. J. Mech. Phys. Solids 23, 371–394.

Runesson, K., Ottosen, N.S., Perić, D., 1991. Discontinuous bifurcation of elastic–plastic solutions at plane stress and plane strain. Int. J. Plasticity 7, 99–121.

Schreyer, H.L., Zhou, S., 1995. A unified approach for predicting material failure and decohesion. In: Ghoniem, N. (Ed.), Plastic and Fracture Instabilities in Materials, AMD-Vol. 200/MD-Vol. 57. ASME, New York, pp. 203–214.

Seregin, G.A., 1984. On the correctness of variational problems of elastic–perfectly plastic continuum mechanics (in Russian). Dokl. Akad. Nauk SSSR 276, 71–75.

Seregin, G.A., 1985. On the formulations of the problems of perfectly elastic–plastic body (in Russian). Prikl. Mat. Mekh. 49, 849–859.

Skrzypek, J., 1993. Plasticity and Creep, Theory, Examples and Problems. Begell House–CRC Press, Boca Raton.

Skrzypek, J., Muc, A., 1988. Limit states of geometrically nonlinear elastic–plastic toroidal shells under pressure with bending. Int. J. Mech. Sci. 30, 347–363.

Skrzypek, J., Życzkowski, M., 1969. Non-symmetrical plastic collapse modes of a symmetrically loaded circular cylinder. Acta Mechanica 7, 213–223.

Skrzypek, J., Życzkowski, M., 1983. Termination of processes of finite plastic deformations of incomplete torroidal shells. Solid Mech. Arch. 8, 39–98.

Storåkers, B., 1971. Bifurcation and instability modes in thick-walled rigid-plastic cylinders under pressure. J. Mech. Phys. Solids 19, 339–353.

Szuwalski, K., 1979. Nośność rozdzielcza pierścieniowej tarczy kołowo-symetrycznej ze sztywną inkluzją. Mech. Teor. Stosowana 17, 589–602.

Szuwalski, K., 1990. Decohesive carrying capacity in perfect and asymptotically perfect plasticity (a survey). Mech Teor. Stosowana 28, 243–253.

Szuwalski, K., Życzkowski, M., 1984. A concave interaction curve corresponding to decohesive carrying capacity under thermal loadings. J. Thermal Stresses 7, 121–134.

Temam, R., Strang, G., 1980. Functions of bounded deformation. Arch. Ration. Mech. and Analysis 75, 7–21.

Thomas, T.Y. Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics. J. Math. Mechanics 6, 311–322, 907–908.

Thomas, T.Y., 1961. Plastic Flow and Fracture in Solids. Academic Press, New York, London.

Tvergaard, V., 1978. On the burst strength and necking behaviour of rotating disks. Int. J. Mech. Sci. 20, 109–120.

Życzkowski, M., 1958. Certain general formulae for plane circularly symmetric plastic states. Arch. Mech. Stos. 10, 463–478.

Życzkowski, M., 1965. Operations on generalized power series. Z. Angew. Math. Mechanik 45, 235–244.

Życzkowski, M., 1981. Combined Loadings in the Theory of Plasticity. PWN-Nijhoff, Warszawa-Alphen aan den Rijn.

Życzkowski, M., 1999. Discontinuous bifurcations in the case of the Burzyński-Torre yield condition. Acta Mechanica 132, 19–35.

Życzkowski, M., Szuwalski, K., 1982. On the termination of the process of finite plastic deformations. J. Méc. Theor. Appliquée, Numero special I, 175–186.

Życzkowski, M., Tran, L.B., 1997. Interaction curves corresponding to the decohesive carrying capacity of a cylindrical shell under combined loading. Int. J. Plasticity 13, 551–570.

Życzkowski, M., Szuwalski, K., Skoczeń, B., 1992. Termination of process of finite plastic deformations in disks under combined loadings. In: Besdo, Stein (Eds.), Proc. IUTAM Symp. Finite Inelastic Deformations. Springer, pp. 547–556.